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DYNAMICS OF CAPILLARY WAVES ON A BUBBLE PERFORMING NONLINEAR PULSATIONS IN A LOW-VISCOSITY LIQUID

O. V. Voinov

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We examine the dynamics of capillary waves of small amplitude on a bubble, performing initially spherosymmetric pulsations in a liquid of low viscosity. We study in the shortwave approximation the characteristics of the growth of surface disturbances with arbitrary pressure differentials and polytropic exponents. We determine the asymptotic time dependences of the mean-square amplitude of the disturbances and obtain approximate formulas for the amplitude growth characteristics for a wide range of parameters. The existence of a single universal dependence of the wave index is discovered. A strong influence of the polytropic exponent near the isotherm on the wave growth characteristics is found. We establish an analogy between the growth of inertial-capillary waves on the surface of a nonlinearly pulsating bubble and on a plane surface with constant acceleration. It is shown that in the case of large-amplitude pulsations the low viscosity approximation is capable of describing the nonsmall effects of change of the wave growth characteristics. We determine the influence of viscosity on the dynamics of the disturbances, and note the viscosity-associated stratification of the universal relation for the wave growth characteristic.

1. Asymptotic Description of Nonstationary Short Waves on a Pulsating Bubble. We shall examine the small perturbations of the spherically symmetric pulsations of a gas bubble in a liquid that is at rest at infinity. We denote the initial gas pressure by p_0 and the pressure at infinity by p_∞. The dynamics of the disturbances on the pulsating gas bubble depends on the polytropic exponent k and the pressure ratio parameter ε and the capillarity parameter σ:

$$\varepsilon = p_0/p_\infty,\ \sigma = \sigma'/p_\infty R_0',\ \sigma_0 = \sigma/\varepsilon$$

(σ' is the surface tension coefficient, R_0' is the initial radius, the primes indicate dimensional quantities). We shall introduce the viscosity influence parameter later, after accounting for the inertial-capillary effects in the ideal fluid formulation.

We examine the limit $\sigma << 1$, when the capillary forces do not influence the change of the bubble radius, defined in dimensionless form by the equation

$$R^{\cdot\cdot}R + (3/2)R^{\cdot 2} = \varepsilon R^{-3k} - 1,\ R(0) = 1,\ R^{\cdot}(0) = 0,$$
$$R = R'/R_0',\ t = t'(p_\infty/\rho)^{1/2}/R_0'. \tag{1.1}$$

The amplitudes of the small perturbations of the surface, represented by a series in the spherical harmonics

$$r = R + \sum_n a_n Y_n(\theta, \varphi),$$

are found for $n >> 1$ from the equation

$$y^{\cdot\cdot} + \lambda^2 q(t)y = 0,\ q = -R^{\cdot\cdot}/R + \sigma n^2/R^3,\ \lambda^2 = n,\ y = a_n R^{3/2}, \tag{1.2}$$

which remains valid even with account for the viscosity of the liquid, although it is necessary to give a different definition to the modified amplitude y (Section 7).

Using the expressions for the linearly independent solutions of the equation (1.2) with $t < T/2$ (T is the pulsation period) and the formula for the characteristic index μ_1, obtained in [1] on the basis of Floquet theory and the WKB-approxi-

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mation, we write the necessary (for analysis of the breakdown of the bubble surface) general solution of the equation (1.2) with accuracy to a constant multiplier in the form

$$\begin{gathered} 0 \leq t < t_* - \delta,\ y = (1/q(t))^{1/4}\cos(\lambda L(t) + \varphi), \\ t_* + \delta < t < t_{*2} - \delta,\ y = (-1/q(t))^{1/4}\cos(\lambda L(t_*) + \varphi + \pi/4)\exp(\lambda K(t)), \\ t_{*2} + \delta < t \leq T,\ y = (1/q(t))^{1/4}\cos(\lambda L(t - T) + \varphi)\exp(\mu_1 T), \end{gathered} \tag{1.3}$$

$$L(t) = \int_0^t \sqrt{q}\,dt,\ K(t) = \int_{t_*}^t \sqrt{-q}\,dt;$$

$$\exp(\mu_1 T) = 2\exp(2\lambda K_0)\cos(2\lambda l_0),\ K_0 = K(T/2),\ l_0 = L(t_*). \tag{1.4}$$

Here t_* is the smallest simple zero of $q(t)(q(t_*) = 0)$; t_{*2} is the second simple zero of $q(t)(t_{*2} = T - t_*)$; φ is an arbitrary constant; $\delta \to 0$; $\lambda \to \infty$. In the analysis of the dynamics of the disturbances the initial phase φ is usually not known. Formulas (1.3) make it possible to exclude the unknown (in principle) parameter φ if we convert to the mean-square amplitude $\langle y^2\rangle$. By virtue of the real randomness of the initial conditions for the disturbances, we can examine the phase φ as an independent random quantity and perform averaging with respect to it. This procedure is logical, specifically, in the region of large values of the wave indices n. After averaging with respect to φ in the interval $[0, \pi]$, there follows from (1.3),(1.4)

$$\begin{gathered} \sqrt{q(t)}\,\langle y^2\rangle = 1,\ 0 \leq t < t_* - \delta, \\ \sqrt{-q(t)}\,\langle y^2\rangle = \exp(2\lambda K(t)),\ t_* + \delta < t < t_{*2} - \delta, \\ \sqrt{q(t)}\,\langle y^2\rangle = \exp(2\mu_1 T),\ t_{*2} + \delta < t \leq T. \end{gathered} \tag{1.5}$$

The first formula (1.5) corresponds to cavitation bubble instability [2], the second corresponds to Rayleigh–Taylor instability [3]; the formulas (1.5) also describe parametric instability of the bubble surface. In addition to these three familiar instability types, in practice the Kelvin–Helmholtz instability [4] may be important, but account for it requires a separate analysis, since it is associated with violation of the spherical symmetry of the flow in the undisturbed state. The fundamental difference between exponential instability of the bubble in the narrow time interval $(t_*, T - t_*)$ and parametric instability is clearly seen from the formulas (1.5). For an arbitrarily large value of $4\lambda K_0 >> 1$, when within the cycle the amplitude certainly exceeds all the allowable limits for small disturbances, we can indicate those values of $2\lambda L(t_*) \sim (l + 1/2)\pi$ (l = 0, 1, ...), for which $\mu_1 \sim 0.$, i.e., parametric instability does not exist. On the other hand, in the case of parametric instability, if it exists, there is present the effect of accumulation of exponential instability in each cycle [1], and failure may be caused by this accumulation.

Formulas (1.5) make it possible to construct a theory of the time of destruction of the surface of the bubble by short waves if we find the dependences of the coefficients on the following parameters: pressure differential $1/\varepsilon$, polytropic exponent k, capillarity σ and describe the initial conditions for $< y^2 >$. Determination of the moment of failure with an accuracy greater than half the interval of exponential growth of the wave $\tau = T/2 - t_*$ is not of interest, therefore it is sufficient to limit ourselves to the values of K(t) for $t = T/2$ and $t = T - t_*$.

In contrast with [1], in addition to the limit $\varepsilon \to 0$ we shall examine the (more difficult for description) region of moderate pressure differentials $p_\infty/p_0 = 1/\varepsilon \lesssim 10$. For validity of the short-wave approximation it is necessary to require simultaneous smallness of both parameters: σ and ε, although for $\varepsilon \to 0$ the maximal wave growth rate increases and the corresponding index n increases. For the manifestation of a significant influence of the capillary forces only in the short-wave region, it is sufficient to require that $\sigma << 1$.

2. Technique for Determining the Indexes n of Growing Waves with Account for the Capillary Forces. For n >> 1 the characteristic index μ_1 depends basically on the magnitude of the exponent — the growth index $2\sqrt{n}K_0$ in (1.4), if the pre-exponential factor is not small in comparison with unity, and for the wave spectrum it can be small only for individual values of the index n. We write the index of growth of the amplitude up to the middle of the cycle as

$$H = \sqrt{n}K_0 = \sqrt{n}\int_{t_*}^{t_m} \sqrt{-q}\,dt,\ \ q(t_*) = 0,\ R^{\cdot}(t_m) = 0. \tag{2.1}$$

Introducing the variable

$$x = \frac{\alpha}{\alpha + 1} R^{-3(k-1)},\ \alpha = \frac{\varepsilon}{k-1}, \tag{2.2}$$

we represent the acceleration and the velocity in the form

$$\frac{R^4 \ddot{R}}{1+\alpha} = kx - 1, \quad \frac{3}{2}\frac{\dot{R}^2}{1+\alpha} = 1 - x - \frac{(1+\alpha)^{-1}}{[(1+1/\alpha)x]^{1/(k-1)}} = f(x). \tag{2.3}$$

In accordance with (2.2) and (2.3), the minimal radius R_m is described by the formula

$$R_m^3 = [(1+1/\alpha)x_m]^{-1/(k-1)} = (1-x_m)(1+\alpha), \tag{2.4}$$

where there appears the parameter x_m — the largest root of the equation

$$f(x_m) = 0. \tag{2.5}$$

The formula

$$\varepsilon = (k-1)(1-R_m^3)/(R_m^{-3(k-1)} - 1), \tag{2.6}$$

holds, and therefore it is not necessary to solve (2.5), we can specify $R_m(\varepsilon)$ parametrically.

We shall indicate in explicit form the relation $x_m(\varepsilon)$ for $\varepsilon \to 0$:

$$\begin{gathered}(1-x_m)/A = 1 + A_1 + (1+k/2)A_1^2 \\ + (1+k/3)(1+k)A_1^3 + \ldots, \end{gathered} \tag{2.7}$$

$$A = (1+\alpha)^{-1}(1+1/\alpha)^{-1/(k-1)}, \quad A_1 = A/(k-1), \quad \alpha = \varepsilon/(k-1).$$

We shall transform K_0 in (2.1), using the notations [1]

$$\Sigma = n^2 b^{-2}\sigma(1+\alpha)^{-1}, \; b = (1+1/\alpha)^{1/3(k-1)}. \tag{2.8}$$

We note that in [1] there is an insignificant as $\varepsilon \to 0$ misprint — the exponent -1 of $(1+\alpha)$ is omitted. In accordance with (2.1)-(2.3) and (2.8)

$$K_0 = \frac{1}{\sqrt{6}(k-1)} \sum_{x_*}^{x_m} \sqrt{\frac{Q}{f}}\frac{dx}{x}, \quad Q = kx - 1 - \Sigma x^{-\frac{2}{3(k-1)}}; \tag{2.9}$$

$$Q(x_*) = 0, \; f(x_m) = 0. \tag{2.10}$$

In place of x it is also convenient to use the variable

$$y = x/x_m, \quad y \in [1/kx_m, 1]. \tag{2.11}$$

Because of the singularity for $k \to 1$, the formula (2.9) is poor for calculations, moreover the quantities Σ and x_* vary strongly for different k or ε. More effective is the technique for determining the index n in which we use as the basic parameters those whose range of variation is independent of ε and k. Therefore we can introduce the new parameters θ and ψ. whose values lie in the interval (0, 1):

$$\begin{gathered}\theta(kx_m - 1) = Q(x_m), \; x_m^{-2/3(k-1)}\Sigma = (1-\theta)(kx_m - 1), \\ y_* = 1/kx_m + \psi(1 - 1/kx_m). \end{gathered} \tag{2.12}$$

In view of $Q(x_*) = 0$, the parameter θ is connected with ψ:

$$1 - \theta = \psi y_*^{2/3(k-1)} = \psi[1 - (1-\psi)(1 - 1/kx_m)]^{2/3(k-1)}. \tag{2.13}$$

From (2.8), (2.9), and (2.12) there follows

$$n^2\sigma R_m^2 = (1-\theta)(kx_m - 1)(1+\alpha). \tag{2.14}$$

Considering (2.2) and the formulas

$$\sigma = \sigma_0 \varepsilon = \sigma_0 \alpha (k - 1),\ \sigma_0 = \sigma' / p_0 R_0',$$

we can transform (2.14) to the form

$$n^2 \sigma_0 = R_m^{1-3k}(1 - \theta) B,$$
$$B = (1 - 1/kx_m)/(1 - 1/k) = (3/2)k\beta. \tag{2.15}$$

The new parameter β is indicated in (2.15), since it appears later in the basic formulas.

3. Analytic Description of Wave Growth Index $K_0(\varepsilon, k, \sigma)$ for Moderate Pressure Differentials $1/\varepsilon$. The integral K_0 (2.9) was studied in [1] for $\varepsilon \to 0$ and individual values of k. Of interest is the region of moderate values of $1/\varepsilon$ and arbitrary $k < 1.7$.

Analytic determination of (2.9) with account for (2.10) is possible in the entire range $0 < \varepsilon < 1$ if we use some expansions with respect to the parameter $1 - y$, which is relatively small. For f in (2.9) it is sufficient to take no more than three terms (as $\varepsilon \to 0$, only one) of the expansion

$$f = \sum_{m=1}^{\infty} b_m (1 - y)^m,\ b_1 = (kx_m - 1)/(k - 1). \tag{3.1}$$

This expansion is ineffective for Q, since $Q(y_*) = 0$, and in (2.9) there appears $\sqrt{Q}$. In place of (3.1) we write

$$Q(y) = (y - y_*) \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dy^n} \left(\frac{Q}{y - y_*} \right) \Big|_{y=1} (y - 1)^n. \tag{3.2}$$

The linear interpolation of Q in the interval $(y_*, 1)$ that was used in [1] corresponds to the first term of the series (3.2).

With the aid of (3.1), (3.2), the integral (2.9) is reduced to one that can be calculated in terms of the elementary functions:

$$K_0 = \frac{1}{\sqrt{6}(k - 1)} \int_{y_*}^{1} \sqrt{\frac{Q(1)(y - y_*)}{(1 - y_*) b_1 (1 - y)}} (1 + \zeta_1 (y - 1)$$
$$+ \zeta_2 (y - 1)^2 + \ldots) \frac{dy}{y} \tag{3.3}$$

(ζ_1, ζ_2 are constants). As a result we obtain from (2.9), (3.1)-(3.3)

$$K_0 = \frac{\pi}{2} \frac{\sqrt{\beta\theta(1 - \psi)}}{1 + \sqrt{y_*}} F,\ F = 1 + \frac{1}{2}(B_0 + a)(1 - \psi)$$
$$+ \frac{1}{4}(1 - \psi)^2 \left(a\left(3a + \frac{\beta}{2} + \frac{7}{6}\Delta\right) + B_0(\beta + 2{,}5\Delta + 2a) \right), \tag{3.4}$$
$$\Delta = 1 - 1/kx_m,\ a = (1/8)(1/x_m - 1)/(k - 1),$$
$$B_0 = (\beta/8)(\beta + \Delta)(1/\theta - 1)(1 - \psi),$$

where the function $F(\psi)$ does not differ strongly from unity. For $F = 1$ and $x_m = 1$, (3.4) agrees with the approximate expression for K_0 with account for σ as $\varepsilon \to 0$, found in [1].

Most important are the values of K_0 for the wave of maximal growth, which corresponds for the middle of the cycle to maxH with $R = R_m$. This maximum also corresponds approximately to the maximum of the coefficient of increase of the amplitude over several cycles if $|\cos(2\lambda I_0)| \sim 1$, $2H >> 1$. The maximum of H with respect to the index n is determined by the conditions of a maximum of $\Sigma^{1/4} K_0$ with respect to ψ, which yields the equation

$$\Omega(\psi) = \Omega_0 + 2\frac{dF}{d\psi} = 0,\ \Omega_0 = \left(\frac{1}{\psi} + \frac{\beta}{y_*}\right)\left(\frac{3}{2} - \frac{1}{\theta}\right) - \frac{1}{(1 - y_*)\sqrt{y_*}}. \tag{3.5}$$

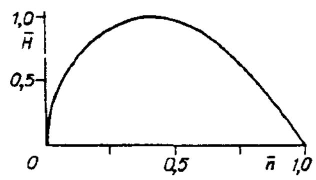

Fig. 1

Here and in the following for brevity we indicate the values of ψ, θ, y_* at the point maxH without special indices; the values at the other points are specified.

We note that the corresponding contribution of F in (3.5) does not exceed 10% of the terms in Ω_0, even for $\varepsilon = 0.8$; moreover, it varies little for different ε and k.

4. Universal Relations for the Growth Index H. Of interest is the derivation of simple formulas that can describe with adequate accuracy K_0, ψ, $1 - \theta$ at the point maxH as functions of k and ε. The parameters ψ and $1 - \theta$ depend weakly on k, in contrast with the parameters that are calculated in terms of the y_* and Σ for maxH, which depend strongly on k. This is the advantage of the variables ψ and $1 - \theta$. The maximal deviation from the average value in the range k = 1 to 1.6 is about 6% for ψ and K_0, while for $1 - \theta$ it is only 1.4%, if $\varepsilon = 0$. This opens up the possiblity of simple approximations. We can represent ψ, $1 - \theta$, and K_0 well by forms that are linear in k or with error much less than 1% by the following expressions for $\varepsilon \to 0$:

$$\psi = 0{,}2666k^{-0{,}3}, \ 1 - \theta = 0{,}1636k^{-0{,}06}, \ K_0 = 0{,}504k^{-0{,}27}. \tag{4.1}$$

It is not difficult to find a more general approximation of K_0, suitable for moderate pressure differentials $1/\varepsilon$, if we note that in (3.4) at the point maxH the factor β varies most strongly with variation of ε and k. Therefore we can seek an approximation of the form $K_0 \sim f(\beta, k)\sqrt{\beta}$, where f is a form that is linear in β. The formula for K_0 for the entire range of pressure differentials $1/\varepsilon = 1$-∞ is found in this way:

$$K_0 = (a_1 + a_2\beta)\sqrt{\beta}, \ a_1 = 0{,}673, \ a_2 = 0{,}204k - 0{,}295. \tag{4.2}$$

The accuracy of (4.2) is no less than $2.4 \cdot 10^{-3}$ for $k \geq 1.1$ and no less than 10^{-2} for k = 1. For $\varepsilon \to 0$ the formula (4.2) is not inferior to the corresponding formula (4.1). For small pressure differentials ($\varepsilon \to 1$, $\beta \to 0$), (4.2) reduces to the form

$$K_0 = 0{,}550\sqrt{(1 - \varepsilon)/k}, \tag{4.3}$$

which differs very little from the exact formula of the linear theory, including the coefficient 0.551.

Calculations show that the parameters $1 - \theta$ and ψ at the point maxH are approximated well by the formula

$$F(k, \varepsilon) = F(k, 0) + (a_1 + a_2(k - 1))(k - kx_m)/(k - 1). \tag{4.4}$$

For $F = 1 - \theta$ $a_1 = 0.0238$, $a_2 = 0.005$, while for $F = \psi$ $a_1 = -0.075$, $a_2 = 0.064$. The accuracy of the formula for $1 - \theta$ is better than 1% for any ε, if $1.1 \leq k \leq 1.6$. For ψ the accuracy is better than 1% for $\varepsilon < 0.8$.

The complete study made of the parameters of the point maxH is of particular importance in connection with the unusual possibility of obtaining (thanks to this) an approximate description of the entire curve H(n) as a whole as a function of ε and k. We introduce the normalized quantities

$$\bar{H} = H/H_{\max}, \ \bar{n} = n/n_0, \tag{4.5}$$

where

$$n_0 = b((1 + \alpha)(k - 1)/\sigma)^{1/2} \tag{4.6}$$

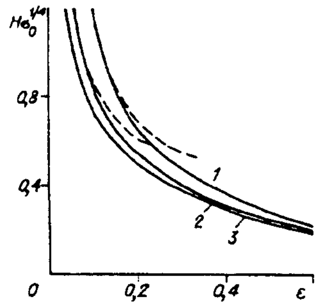

Fig. 2

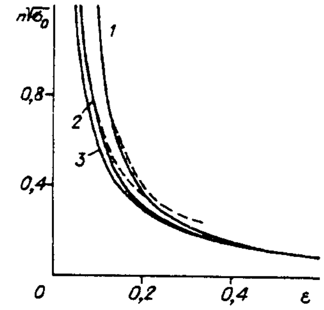

Fig. 3

corresponds to H = 0. The relation $\bar{H}(\bar{n})$ is shown in Fig. 1 for $\varepsilon = 0$ and k = 1.4. The analytic estimates for $\bar{n} \to 0$ and the numerical calculations show that the variables ε and k in the ranges $1/\varepsilon = 2.5$-∞ and $1.1 \le k \le 1.66$ have practically no influence on $\bar{H}(\bar{n})$, which can be considered to be universal. For $\varepsilon = 0$ the maximal deviation of the values of $\bar{H}(\bar{n})$ from the case k = 1.4 is about 0.5%, while for $\varepsilon = 0.4$ it is about 1.5-2%. The universal approximate relation (4.5) for $\bar{H}(\bar{n})$ effectively solves the problem of simple representation of the function of three variables H(ε, k, n).

5. Influence of the Pressure Differential 1/ε and the Index k on the Parameters of the Wave of Maximal Growth. The dependences of maxH and the wave index n on ε are given in Figs. 2 and 3 for k = 1.1, 1.4, 1.6 (curves 1-3) in accordance with the formulas of Sections 3 and 4. For comparison the asymptotic relations for $\varepsilon \to 0$ are shown by the dashed curves. We see that maxH and the corresponding wave index n for $\varepsilon > 0.15$ differ significantly from the asymptotic relations.

We see from Figs. 2 and 3 sharp increase of n and maxH with increase of the pressure differential $1/\varepsilon$, which agrees with [1]. We note a significant new characteristic: according to the graphs, the polytropic exponent k in the region k ~ 1 to 1.2, close to the isotherm k = 1, has a strong influence on the wave index and the values of maxH for $\varepsilon \lesssim 0.2$. This can be explained through the significant influence on the minimal radius $R_m(\varepsilon)$, which is reflected in Fig. 4, where the curves 1-4 correspond to k = 1.0, 1.1, 1.4, 1.6.

For explanation we note the quite simple relations that are associated with the approximations of the form $(1-\theta)k\beta \sim 1-\varepsilon$, $K_0 \sim \sqrt{(1-\varepsilon)/k}$, which are reasonable for moderate values of $1/\varepsilon$ and for $\varepsilon \to 1$. We obtain from (2.15) and (2.1), respectively,

$$n\sqrt{\sigma_0} = 0{,}42R_m^{-(3k-1)/2}\sqrt{1-\varepsilon},$$
$$\sigma_0^{1/4}H = 0{,}37R_m^{-(3k-1)/4}(1-\varepsilon)^{3/4}k^{-1/2}. \qquad (5.1)$$

Writing the formulas (5.1) in the form n = 0.42X and H = 0.37Y, we represent the ratios of the exact values of n and H to X and Y in Fig. 5 for k = 1.1, 1.6 (lines 1, 2). We see that the ratios n/X and H/Y are close to the indicated constants for $\varepsilon > 0.1$. The error of the first formula (5.1) in the region $\varepsilon < 0.4$; $1.1 \le k \le 1.6$ does not exceed 5%. We note that for $\varepsilon \to 0$ the value of n/X differs little from 0.40.

6. Analogy with the Instability with Constant Acceleration. We introduce the Bond number Bo, defined in terms of the wavenumber k_w and the maximal acceleration g_m' for $R = R_m$:

$$\mathrm{Bo} = \frac{\rho g_m'}{\sigma' k_w^2}, \quad g_m' = \ddot{R}_m \frac{p_\infty}{\rho R_0'}, \quad k_w = \frac{n}{R_m R_0'}.$$

From which with account for (2.3), (2.14) we obtain

$$\mathrm{Bo} = (1-\theta)^{-1}, \quad 0 < \theta < 1. \qquad (6.1)$$

For the value of $1-\theta$ at the point maxH, Bo has interesting characteristics: it depends weakly on k and ε. Thus, for $\varepsilon = 0$, Bo = 6.12 to 6.3 with k = 1 to 1.6; for k = 1.4 Bo = 6.23 with $\varepsilon = 0$, and Bo = 5.83 with $\varepsilon = 0.4$. Thus, the value of Bo of the wave of maximal growth is approximately constant with variation of ε and k.

It is advisable to introduce the second Bond number $\overline{\mathrm{Bo}}$, defined on the basis of the average acceleration in the time interval where the boundary acceleration $\ddot{R} > 0$. If we consider that for $\varepsilon \to 0$ and k = 1.4 the average acceleration $\bar{g}' \approx 0.45\, g_m'$, then from (6.1) there follows

$$\overline{\mathrm{Bo}} = \rho\bar{g}'/\sigma' k_w^2 = 2{,}8. \tag{6.2}$$

We shall compare the value of (6.2) with the corresponding Bo number in the case of constant acceleration of a plane boundary. In this case the wavenumber k_w is found from the condition of a maximum of the difference $k_w\bar{g}' - k_w^3\sigma'/\rho$ and Bo = 3. This means that Bo based on the average acceleration of the bubble surface is close to Bo with constant acceleration of a plane boundary, i.e., there is a definite analogy between the two different problems.

It is also interesting to compare the maximal indices of the growth of the amplitude of the disturbances in the indicated cases. For $g' = \mathrm{const}$, the maximal index of growth over the time τ has the form

$$H = \tau\sqrt{(2/3)k_w g'}.$$

We shall calculate H as a function of τ and the average g' for $\varepsilon \to 0$. We write the time 2τ, during which $R^{\cdot\cdot} > 0$, for k = 1.4 approximately as

$$2\tau = 3{,}1 R_m^{5/2} R_0' \sqrt{\rho/p_\infty}.$$

Hence, considering that $\bar{g}' = 0.45\, g_m'$, we find $H = 0.54\sqrt{n}$, $K_0 = 0.54$. This differs very little from $K_0 = 0.461$, which the exact calculation yields for $\varepsilon \to 0$.

Thus, with regard to the Bond number and with regard to the index of wave growth with strongly varying acceleration of the bubble surface near the minimal radius there is an analogy with the very simple case of Taylor instability with constant acceleration of a phase interface. This analogy is useful for approximate estimates and can serve as confirmation of the reasonableness of the calculations performed.

7. Model of Influence of Liquid Viscosity. We shall examine the case of small viscosity, when its influence can be taken into account by the introduction of corrections to the equations for the amplitudes of the disturbances. The viscosity is small if the relative rate of change of the amplitude of a short wave (n >> 1) is sufficiently large:

$$\lambda\sqrt{|q|} \gg 2\nu n^2/R'^2 = 2\nu k_w^2,\ \lambda = \sqrt{n} \tag{7.1}$$

(ν is the kinematic viscosity).

For a wave with the frequency ω the condition (7.1) has the very simple form $\omega >> 2\nu k_w^2$. In order to account for viscous decay in the equation for the amplitudes, it is sufficient to consider the elementary solution of the problem of the decay of a capillary wave on a spherical bubble in a spherically symmetric mass force field or include in the energy balance equation the corresponding expression for the energy dissipation in a potential velocity field. We write the equations of the dynamics of the amplitudes with account for small viscosity in dimensionless notations in the form

$$\begin{gathered} Ra_n^{\cdot\cdot} + 3R^{\cdot}a_n^{\cdot} - (n-1)a_nR^{\cdot\cdot} + \sigma n^3R^{-2}a_n = -4\nu_1 k_w^2 Ra_n^{\cdot}, \\ \nu_1 = (\nu/R_0')\sqrt{\rho/p_\infty},\ a_n = a_n'/R_0',\ k_w = n/R. \end{gathered} \tag{7.2}$$

The transformation

$$a_n = \frac{y}{R^{3/2}}\exp\left(-2\nu_1 n^2\int_0^t \frac{dt}{R^2}\right) \tag{7.3}$$

converts (7.2) to the equation

$$\begin{gathered} y^{\cdot\cdot} + [-(n+1/2)R^{\cdot\cdot}/R + \sigma n^3/R^3 - \delta_0^2]y = 0, \\ \delta_0^2 = (2\nu_1 k_w^2)^2 + 2\nu_1 k_w^2 R^{\cdot}/R - (3/4)(R^{\cdot}/R)^2. \end{gathered} \tag{7.4}$$

For n >> 1, of the three terms in δ_0^2 it is sufficient to consider only the first. Equation (7.4) differs little from (1.2) if with account for the principal terms with respect to n the following condition is satisfied

$$|-nR^{\cdot\cdot}/R + \sigma n^3/R^3| \gg (2\nu_1 k_w^2)^2, \tag{7.5}$$

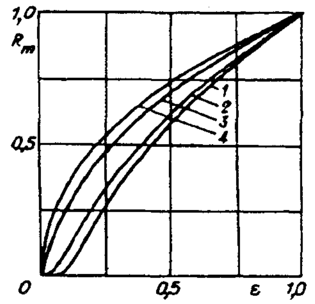

Fig. 4

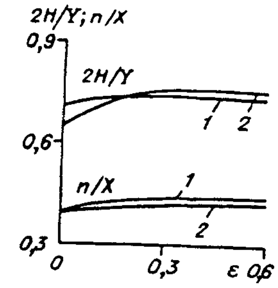

Fig. 5

which corresponds to smallness of the frequency shift $|\omega|^2 >> \delta_0^2$ or smallness of the decay δ_0. We can use the asymptotic solution (1.3) for determination of the quantity y(t), connected with the amplitude a by the formula (7.3). For the asymptotic solution (1.3) the inequality (7.5) agrees with (7.1).

It is of interest to examine the inequality (7.5) at each moment of the pulsation. It is evident that it is always violated near the zeros of $q(t) = 0$, $t = t_*$, $t = T - t_*$. Far from the zero of q with $R >> R_*$, we rewrite (7.5) with account for $R^{\cdot} \approx -R^{-4}$:

$$R^{-1} + \sigma R n^2 \gg 4\nu_1^2 n^3. \tag{7.6}$$

On the other side of the zero of q for $R < R_*$ it is best to represent the inequality (7.5) or (7.1) in some average sense for the interval $(t_*, T/2)$, where it is advisable to take n to mean the index of the wave of maximal growth. Then, using the relation (6.2) between the average acceleration and the characteristic wavenumber k_w, we replace approximately the left side of (7.5) by $2\sigma n^2/R_m^3$, writing

$$\sigma R_m \gg 2\nu_1^2 n. \tag{7.7}$$

The first of the inequalities (7.6) and (7.7) is no more restrictive than the second. Therefore it is sufficient to require the satisfaction of (7.7).

Formula (7.3) makes it possible to determine the influence of viscosity on the change of the amplitude of the disturbance. For half of the cycle it is sufficient to calculate the integral

$$I_\mu(\varepsilon, k) = \int_0^{T/2} \frac{dt}{R^2}, \tag{7.8}$$

and express the contribution of viscosity to change of the disturbance amplitude a_n by the coefficient exp (ΔH), where

$$\Delta H = -2I_\mu \nu_1 n^2. \tag{7.9}$$

With increase of the pressure differential $(1/\varepsilon \to \infty)$ the coefficient I_μ is a limited quantity, and there exists the limit of the integral (7.8):

$$\lim_{\varepsilon \to 0} \int_0^{T/2} \frac{dt}{R^2} = \sqrt{\frac{3}{2}} \int_0^1 \frac{dR}{R^2\sqrt{R^{-3} - 1}} = 2{,}975. \tag{7.10}$$

The fact of convergence of the integral (7.10) for $R = 0$ indicates that the primary influence of viscosity on the change of the amplitude for $\varepsilon \to 0$ is concentrated in the region of large values of the radius, far from the zone of exponential growth of the disturbances $R < R_*$.

The dependence (7.8) of the viscous decay coefficient I_μ on ε for $k = 1.4$ and 1.66 (lines 1, 2) is shown in Fig. 6.

The effectiveness of the viscosity influence model (7.8), (7.9) is determined by the following question: is a finite (not small) effect of the influence of viscosity on the disturbance growth index H, i.e., $|\Delta H| \sim H$, possible with retention of the conditions of validity of the small viscosity approximation? It is interesting therefore to evaluate the maximal ratios

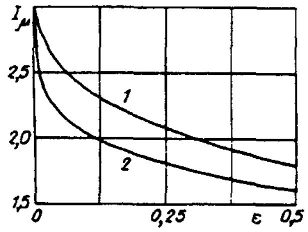

Fig. 6

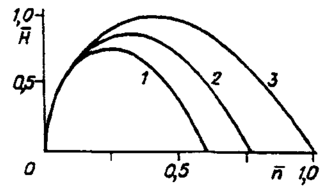

Fig. 7

$|\Delta H|/H$, that are admissible in the framework of the small viscosity approximation. From the formulas (7.9), (5.1) and the approximate relation $\varepsilon \sim (k - 1)R_m^{3(k-1)}$, following from (2.6) for nonsmall $k - 1$, for $\varepsilon \to 0$ we write approximately

$$\frac{|\Delta H|}{\max H} = \frac{2.8\sqrt{k(k-1)}}{R_m^{(3/4)(k+1)}}\sigma_0^{1/4}\mathrm{Ca}, \quad \mathrm{Ca} = \frac{\mu}{\sigma'}\sqrt{\frac{p_0}{\rho}}, \quad \sigma_0 = \frac{\sigma'}{p_0 R_0'}. \tag{7.11}$$

With the use of formulas (5.1) and (7.2), we rewrite the inequality (7.7) in the form

$$\sigma_0^{1/2}\mathrm{Ca}^2 R_m^{-(3k+1)/2} \ll 1. \tag{7.12}$$

Substitution of the expression (7.11) into (7.12) makes it possible to obtain the equivalent to (7.7) condition of validity of the small viscosity approximation for the waves of maximal growth:

$$c_0 R_m(\Delta H/H)^2 \ll 1 \tag{7.13}$$

($c_0 = 0.1$ to 0.25 for $k = 1.6$ to 1.3). The viscosity influence effect is large if $\Delta H/H = 1$, i.e., the increase of the amplitude of the wave of maximal growth is completely suppressed as the result of the pulsation cycle. In this case with $\varepsilon << 1$ the left side of (7.13) contains the small parameter R_m, and therefore the condition (7.13) is satisfied. Consequently, in the framework of the small viscosity approximation a nonsmall effect of the influence of viscosity on the development of the disturbances as a result of the nonlinear bubble pulsation cycle is possible. This fundamentally important conclusion characterizes the primary property of the viscosity influence model (7.8), (7.9). To clarify the obtained result we note that with increase of the pressure differential $1/\varepsilon$ and reduction of R_m the duration of the exponential growth of the disturbances decreases as $R_m^{5/2}$, because of which the contribution of the region of small radii ($R \sim R_m$) to the decay has the order $\sqrt{R_m}$. The contribution to the decay of the remaining region, in which the amplitude of the disturbances does not increase but rather only decreases, increases by $1/\sqrt{R_m}$ in relation to the contribution of the region of small radii. Because of the growth of the relative contribution to the decay of the region of large radii ($R > R_*$), a significant effect of reduction of the amplitude of the disturbances in the small viscosity approximation is possible.

The influence of viscosity on the universal dependence of the index H on the wave index n is defined by the relation

$$\frac{2I_\mu \nu_1 n^2}{\max(\sqrt{n}K_0)} = \mathrm{Ca}\sigma_0^{1/4}\frac{2I_\mu(kx_m - 1)^{3/4}(1+\alpha)^{3/4}}{\varepsilon^{1/4}R_m^{3/2}\max((1-\theta)^{1/4}K_0)}(1-\theta),$$
$$n = n_0(1-\theta)^{1/2}, \quad 0 < \theta < 1. \tag{7.14}$$

Hence we see that the initial pressure p_0 and the initial radius R_0' have very little influence on the influence of the viscosity on the form of the curve H(n) in the universal coordinates (4.5) — only as $(p_0/R_0')^{1/4}$ — but the quantity ε does have a significant influence. The influence of the viscosity on the dependence of H on n, determined with the aid of (7.14), is reflected in Fig. 7 for nitrogen or hydrogen bubbles ($k \approx 1.4$) in water with $\nu = 10^{-2}$ cm²/sec, $R_0' = 0.1$ cm, $p_0 = 0.1$ MPa, $\sigma' = 0.072$ N/m, and also $\varepsilon = 0.1, 0.2$ (lines 1, 2), the curve 3 corresponds to $\nu = 0$. We see that the viscosity leads to stratification with respect to the parameter ε of the universal dependence of H/H_{max} on n/n_0. It is important that the viscosity has a noticeable effect under real conditions of gas bubbles in water. It is evident that change of the viscosity of the liquid in the experiment may serve as a regulator of the dynamics of the disturbances.

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